04/11/2020

venerdì 4 dicembre 2020 14:42

INTEGRALI

PER SOSTITUZIONE: $f: I \to \mathbb{R}$ continua
$\varphi: J \to I \quad \varphi(t) = x$ derivabile

$$\int f(x)\,dx = \int f(\varphi(t)) \cdot \varphi'(t)\,dt$$

es.

- $$\int \cos\underbrace{\left(\frac{2x-5}{3}\right)}\,dx = \frac{3}{2}\int \cos\left(\frac{2x-5}{3}\right)\cdot\frac{2}{3}\,dx = 3x\sin\left(\frac{2x-5}{3}\right)+C$$

Ⓐ $f(x) \quad f'(x) = \frac{2}{3}$

Ⓑ $\frac{2x-5}{3} = t$

$$2x = 3t+5 \qquad x = \frac{3t+5}{2} = \varphi(t)$$

$$\varphi'(t) = \frac{3}{2}$$

$$\int \cos\left(\frac{2x-5}{3}\right) = \int \cos t \cdot \frac{3}{2}\,dt = \frac{3}{2}\sin t + c$$

$$C + \frac{3}{2}\sin t = \frac{3}{2}\sin\left(\frac{2x-5}{3}\right)+C$$

- $$\int \frac{1+\sqrt{x}}{1-\sqrt{x}}\,dx$$

$D_f: \begin{cases} x > 0 \\ 1-\sqrt{x} \neq 0 \to x \neq 1 \end{cases}$

$D_f: (0;1) \cup (1;+\infty)$

$\sqrt{x} = t$

$x = t^2 = \varphi(t) \qquad \varphi'(t) = 2t$

$$\int \frac{1+\sqrt{x}}{1-\sqrt{x}}\,dx = \int \frac{1+t}{1-t}\cdot 2t\,dt = 2\int \frac{t+t^2}{1-t}\,dt =$$

$$\begin{array}{r|l} t^2+t+0 & -t+1 \\ -t^2+t & \overline{-t-2} \\ \hline 2t & \\ -2t+2 & \\ \hline +2 & \end{array}$$

R

$$t^2+t = (1-t)(-t-2)+2$$

$$= 2\int\left(-t-2+\frac{2}{1-t}\right)dt = 2\left[-\int t\,dt - 2\int dt + \int \frac{2}{1-t}\,dt\right] =$$

$$= 2\int\left(-t-2+\frac{2}{1-t}\right)dt = 2\left[-\int t\,dt - 2\int dt + \int\frac{2}{1-t}\,dt\right] =$$

$$= 2\left(-\frac{t^2}{2} - 2t - 2\int\frac{-1}{1-t}\,dt\right) =$$

$$= -t^2 - 4t - 4\ln|1-t| + c \qquad t=\sqrt{x}$$

$$= -x - 4\sqrt{x} - 4\ln|1-\sqrt{x}| + c$$

se $0<x<1$ $\quad F(x) = -x-4\sqrt{x}-4\ln(1-\sqrt{x})+c$

se $x>1$ $\quad F(x) = -x-4\sqrt{x}-4\ln(\sqrt{x}-1)+c$

- $\int\frac{x}{\sqrt{x+3}}\,dx \qquad D_f: x>-3$

$\sqrt{x+3}=t$

$x+3=t^2 \longrightarrow x=t^2-3=\varphi(t) \quad \varphi'(t)=2t$

$$\int \frac{x}{\sqrt{x+3}}\,dx = \int\frac{t^2-3}{\cancel{t}}\cdot 2\cancel{t}\,dt = 2\int t^2-3\,dt = 2t^3-6t+c$$

$$= \frac{2}{3}\sqrt{(x+3)^3} - 6\sqrt{x+3} + c$$

- $\int\frac{e^{2x}}{1+e^x}\,dx$

$e^x=t \quad x=\ln t=\varphi(t) \qquad \varphi'(t)=\frac{1}{t}$

$\boxed{t>0}$

$$\int\frac{e^{2x}}{1+e^x}\,dx = \int\frac{t^{\cancel{2}}}{1+t}\cdot\frac{1}{\cancel{t}}\,dt = \int\frac{t}{1+t}\,dt =$$

$$=\int\left(\frac{t+1}{t+1}-\frac{1}{t+1}\right)dt = \int\left(1-\frac{1}{t+1}\right)dt = t-\ln|t+1|+c=$$

$$=t-\ln(t+1)+c$$

$$=e^x-\ln(e^x+1)+c$$

- $\int\sqrt{\frac{x+3}{x-1}}\,dx$

$\sqrt{\frac{x+3}{x-1}}=t \longrightarrow \frac{x+3}{x-1}=t^2$

$x+3=t^2(x-1)$

$$x+3=t^2(x-1)$$

$$x(1-t^2)=-t^2-3 \qquad x=\frac{t^2+3}{t^2-1}=\varphi(t)$$

$$\varphi'(t)=\frac{2t(t^2-1)-(t^2+3)2t}{(t^2-1)^2}=$$

$$=\frac{2t^3-2t-2t^3-6t}{(t^2-1)^2}=\frac{-8t}{(t^2-1)^2}$$

$$\int\sqrt{\frac{x+3}{x-1}}\,dx=\int t\cdot\frac{-8t}{(t^2-1)^2}\,dt=-8\int\frac{t^2}{(t^2-1)^2}\,dt=$$

$$\int\frac{t^2}{(t^2-1)^2}\,dt=\int\frac{t}{2}\cdot\frac{2t}{(t^2-1)^2}\,dt=\frac{1}{2}\int t\cdot\frac{2t}{(t^2-1)^2}\,dt$$

$$g(t)=t \qquad f'(t)=2t(t^2-1)^{-2}$$

$$g'(t)=1 \qquad f(t)=\frac{(t^2-1)^{-1}}{-1}$$

$$\frac{1}{2}\left[\left(\frac{-t}{t^2-1}\right)-\int\frac{-1}{t^2-1}\,dt\right]=\frac{1}{2}\left(-\frac{t}{t^2-1}\right)+\frac{1}{2}\int\frac{1}{(t+1)(t-1)}\,dt$$

$$\frac{1}{(t+1)(t-1)}=\frac{A}{t+1}+\frac{B}{t-1}$$

$$A(t-1)+B(t+1)=1$$

$$(A+B)t-A+B=1 \qquad \begin{cases}A+B=0\\-A+B=1\end{cases}$$

$$A=\frac{-1}{2}$$

$$B=\frac{1}{2}$$

$$\frac{-t}{2(t^2-2)}+\frac{1}{2}\left[\int\frac{-1/2}{t+1}\,dt+\int\frac{1/2}{t-1}\,dt\right]=$$

$$=\frac{-t}{2(t^2-2)}-\frac{1}{4}\ln|t+1|+\frac{1}{4}\ln|t-1|+c$$

$$\int\sqrt{\frac{x+3}{x-1}}\,dx=\frac{4t}{t^2-1}+2\ln|t+1|-2\ln|t-1|+c$$

$$=\frac{4t}{t^2-1}+2\ln\frac{|t+1|}{|t-1|}+c$$

$$\frac{4\sqrt{\frac{x+3}{x-1}}}{\frac{x+3}{x-1}-1}+2\ln\frac{\sqrt{x+3}+\sqrt{x-1}}{|\sqrt{x+3}-\sqrt{x-1}|}+c$$

$$=(x-1)\sqrt{\frac{x+3}{x-1}}+2\ln\frac{\sqrt{x+3}+\sqrt{x-1}}{|\sqrt{x+3}-\sqrt{x-1}|}+c$$

$$\bullet \int \frac{\ln(8+x\sqrt{x})}{\sqrt{x}} \qquad Df: \left.\begin{matrix} x>0 \\ 8+x\sqrt{x}>0 \end{matrix}\right\} x>0$$

$$\sqrt{x}=t \qquad x=t^2 \qquad \varphi(t)=2t$$

$$\int \frac{\ln(8+x\sqrt{x})}{\sqrt{x}} = \int \frac{\ln(8+t^3)}{\cancel{t}} \cdot 2\cancel{t}\, dt = 2\int \underbrace{\ln(8+t^3)}_{g(t)} dt =$$

per parti

$$= 2\left[t\ln(8+t^3) - \int \frac{3t^3}{8+t^3}\,dt\right] =$$

$$= 2\left[t\ln(8+t^3) - 3\int \frac{t^3+8-8}{t^3+8}\,dt\right] =$$

$$= 2\left[t\ln(8+t^3) - 3\int \left(1-\frac{8}{t^3+8}\right)dt\right] =$$

$$= 2\left[t\ln(8+t^3) - 3t + 24\int \frac{1}{t^3+8}\,dt\right] =$$

$$\int \frac{1}{t^3+8}\,dt = \qquad t^3+8=(t+2)(t^2+4-2t)$$

$$\frac{1}{t^3+8} = \frac{A}{t+2} + \frac{Bt+C}{t^2+4-2t}$$

$$At^2+4A-2At+Bt^2+Ct+2Bt+2C=1$$

$$(A+B)t^2+(2B-2A+C)t+4A+2C=1$$

$$\begin{cases} A+B=0 & A=-B & B=-1/12 \\ 2B-2A+C=0 & 4A=C & A=1/12 \\ 4A+2C=1 & 3C=1 & C=1/3 \end{cases}$$

$$\int \frac{1}{t^3+8}\,dt = \frac{1}{12}\int \frac{1}{t+2}\,dt - \frac{1}{12}\int \frac{-t+4}{t^2+4-2t}\,dt =$$

$$= \frac{1}{12}\ln|t+2| - \frac{1}{12}\int \frac{t-4}{t^2-2t+4}\,dt =$$

$$= \frac{1}{12}\ln|t+2| - \frac{1}{24}\int \frac{2t-8}{t^2-2t+4}\,dt =$$

$$= \frac{1}{12}\ln|t+2| - \frac{1}{24}\int \left(\frac{2t-2}{t^2-2t+4} - \frac{6}{t^2-2t+4}\right)dt =$$

$$= \frac{1}{12}\ln|t+2| - \frac{1}{24}\ln(t^2-2t+4) + \frac{1}{4}\int \frac{dt}{t^2-2t+4}\,dt =$$

$$t^2-2t+4 = (t^2-2t+1)+3 =$$

$$= (t-1)^2+3 = 3\left[1+\frac{(t-1)^2}{3}\right] =$$

$$= 3\left[1+\left(\frac{t-1}{\sqrt3}\right)^2\right]$$

$$= \frac{1}{12}\ln|t+2| - \frac{1}{24}\ln(t^2-2t+4) + \frac{1}{12}\int \frac{1}{1+\left(\frac{t-1}{\sqrt3}\right)^2}\,dt =$$

$$= \frac{1}{12}\ln|t+2| - \frac{1}{24}\ln(t^2-2t+4) + \frac{\sqrt3}{12}\,\mathrm{arctg}\left(\frac{t-1}{\sqrt3}\right)^2 + c =$$

$$2\int \ln(8+t^3)\,dt = 2\left[t\ln(8+t^3) - 3t + 2\ln|t+2| - \right.$$
$$\left. - \ln(t^2-2t+4) + 2\sqrt3\,\mathrm{arctg}\left(\frac{t-1}{\sqrt3}\right)\right] + c =$$

$$= 2\sqrt{x}\ln(8+x\sqrt{x}) - 6\sqrt{x} + 4\ln(\sqrt{x}+2) - 2\ln(x-2\sqrt{x}+4) +$$
$$+ 4\sqrt3\,\mathrm{arctg}\left(\frac{\sqrt{x}-1}{\sqrt3}\right) + c$$

INTEGRALI DEFINITI

$\int_a^b f(x)\,dx$ $\qquad$ f continua $[a,b]$

TEOREMA

$F(x)$ primitiva di $f(x)$ su $[a,b]$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

- $\int_0^2 \ln(x^2+1)\,dx$ $\qquad$ $f(x) = \ln(x^2+1)$ $\quad D_f: \mathbb{R}$ continua

$\int \ln(x^2+1)\,dx$ $\qquad$ PER PARTI

$$= x\ln(x^2+1) - \int \frac{2x^2}{x^2+1}\,dx =$$

$$= x\ln(x^2+1) - 2\int \frac{x^2+1-1}{x^2+1}\,dx =$$

$$= x\ln(x^2+1) - 2\int\left(1 - \frac{1}{x^2+1}\right)dx =$$

$$= x\ln(x^2+1) - 2x + 2\,\mathrm{arctg}(x) + c$$

$$\int_0^2 \ln(x^2+1)\,dx = F(2) - F(0) = 2\ln(5) - 4 + 2\,\mathrm{arctg}(2)$$

- $\int_2^4 \frac{1}{\sqrt{x}(x-1)}\,dx$ $\qquad$ $f(x) = \frac{1}{\sqrt{x}(x-1)}$ $\quad x>0$, $x\neq 1$

$$\int \frac{1}{\sqrt{x}(x-1)}\,dx$$

$$\int \frac{1}{\sqrt{x}(x-1)}dx$$

$\sqrt{x} = t \quad x = t^2 \quad \varphi'(t) = 2t$

$$\int \frac{1}{\sqrt{x}(x-1)}dx = \int \frac{1}{\cancel{t}(t^2-1)} 2\cancel{t}\, dt = 2\int \frac{1}{t^2-1} dt$$

$$\frac{1}{(t-1)(t+1)} = \frac{1/2}{t-1} + \frac{-1/2}{t+1}$$

$$= 2\int \frac{1/2}{t-1} dt + 2\int \frac{-1/2}{t+1} dt =$$

$$= \ln|t-1| - \ln|t+1| + c$$

$$= \ln|\sqrt{x}-1| - \ln|\sqrt{x}+1| + c$$

$$\int_2^4 \frac{1}{\sqrt{x}(x-1)} = F(4) - F(2) =$$

$$= \ln(1) - \ln(3) - \ln(\sqrt{2}-1) + \ln(\sqrt{2}+1)$$

$$= \ln(\sqrt{2}+1) - \ln(\sqrt{2}-1) - \ln(3)$$

OSS. $\sqrt{x} = t \quad x = t^2 \quad \varphi'(t) = 2t$

se $x = 2 \quad t = \sqrt{2}$

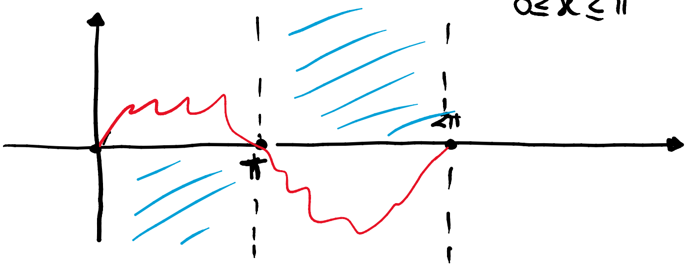
$x = 4 \quad t = 2$

$$2\int_{\sqrt{2}}^{2} \frac{1}{t^2-1} dt$$

- AREA SOTTESA A $f(x) = e^{-2x}\sin(x)$ $[0; 2\pi]$

$f(x) = e^{-2x}\sin(x) \quad f(x) \geq 0$ se $\sin(x) \geq 0$

$0 \leq x \leq \pi$



$$\text{AREA} = \int_0^{\pi} e^{-2x}\sin(x)dx - \int_{\pi}^{2\pi} e^{-2x}\sin(x)dx$$

$$F(x) = \int e^{-2x}\underbrace{\sin(x)}_{g(x)}dx = \text{ per parti}$$

$$= -\frac{1}{2}e^{-2x}\sin x + \int \frac{1}{2}e^{-2x}\underbrace{\cos x}_{g(x)}dx =$$

$$= -\frac{1}{2}e^{-2x}\sin x + \frac{1}{2}\left[-\frac{1}{2}e^{-2x}\cos(x) - \int \frac{1}{2}e^{-2x}\sin x\, dx =\right.$$

$$\ldots \frac{1}{2} e \ldots \int \frac{1}{2} e \ldots \underbrace{\ldots}_{g(x)}$$

$$= -\frac{1}{2}e^{-2x}\sin x + \frac{1}{2}\left[-\frac{1}{2}e^{-2x}\cos(x) - \int \frac{1}{2}e^{-2x}\sin x\,dx\right] =$$

$$= -\frac{1}{2}e^{-2x}\sin x - \frac{1}{4}e^{-2x}\cos x - \frac{1}{4}\int e^{-2x}\sin x\,dx$$

$$\frac{5}{4}\int e^{-2x}\sin x\,dx = -\frac{1}{2}e^{-2x}\sin x - \frac{1}{4}e^{-2x}\cos x$$

$$\int e^{-2x}\sin x\,dx = -\frac{2}{5}e^{-2x}\sin x - \frac{1}{5}e^{2x}\cos x$$

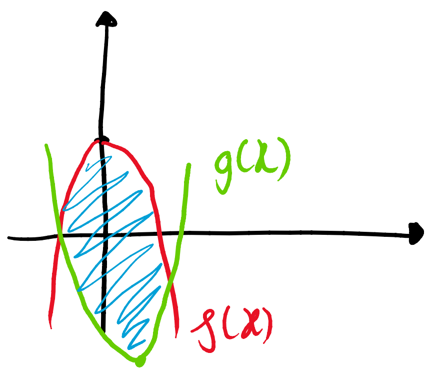
$$A = \int_0^{\pi} f(x)\,dx - \int_{\pi}^{2\pi} f(x)\,dx = [F(\pi)-F(0)] - [F(2\pi)-F(\pi)] =$$

$$= 2F(\pi) - F(0) - F(2\pi) =$$

$$= 2\left(-\frac{1}{5}e^{-2\pi}\cos\pi\right) - \left(-\frac{1}{5}e^{0}\cos 0\right) - \left(-\frac{1}{5}e^{-4\pi}\cos 2\pi\right) =$$

$$= \frac{2}{5}e^{-2\pi} + \frac{1}{5} + \frac{1}{5}e^{-4\pi}$$

- AREA TRA $f(x) = 3 - x^2$ e $g(x) = x^2 - 2x - 3$



$$f(x) = g(x)$$

$$3 - x^2 = x^2 - 2x - 3$$

$$2x^2 - 2x - 6 = 0$$

$$x^2 - x - 3 = 0 \qquad \Delta = 1 + 12 = 13$$

$$x = \frac{1 \pm \sqrt{13}}{2} \rightarrow a,\ b$$

$$A = \int_a^b |f(x) - g(x)|\,dx =$$

$$= \int_a^b |3x^2 - x^2 + 2x + 3|\,dx = \int_a^b |6 - 2x^2 + 2x|\,dx =$$

$$= 2\int_a^b |3 - x^2 + x|\,dx$$

$$-x^2 + x + 3 > 0 \quad \text{se} \quad a < x < b$$

$$-x^2 + x + 3 < 0 \quad \text{se} \quad x < a \vee x > b$$

$$= \int_a^b (-x^2 + x + 3)\,dx = \left[-\frac{1}{3}x^3 + \frac{x^2}{2} + 3x\right]_a^b$$

INTEGRALI IMPROPRI

$\int_a^b f(x)\,dx$ $f(x)$ non è definita in $x=a \vee x=b$ o $\forall$ punto dell'intervallo

NON vale il teorema fondamentale del calcolo integrale

- $\int_{-1}^{7} \frac{1}{x}\,dx =$ $f(x)=\frac{1}{x}$ $D_f: x\neq 0$

$f(x)$ non è continua

$$= \int_{-1}^{0} \frac{1}{x}\,dx + \int_{0}^{7} \frac{1}{x}\,dx =$$

$$= \lim_{\varepsilon\to 0} \int_{-1}^{-\varepsilon} \frac{1}{x}\,dx + \lim_{\delta\to 0} \int_{\delta}^{7} \frac{1}{x}\,dx =$$

$$\lim_{\varepsilon\to 0} \ln|-\varepsilon| - \ln|-1| = -\infty$$ DIVERGE

- $\int_a^{+\infty} f(x)\,dx$ anche qua NON VALE il teorema del calcolo d'integrale

- $\int_3^{+\infty} \frac{x-1}{x+4}\,dx$ $D_f: x\neq -4$

$$\lim_{M\to+\infty} \int_3^M \frac{x-1}{x+4}\,dx$$

$$F(x) = \int \frac{x-1}{x+4}\,dx = \int \frac{x+4-5}{x+4}\,dx = \int 1 - \frac{5}{x+4}\,dx =$$

$$= x - 5\ln|x+4| + C$$

$$\lim_{M\to+\infty} F(M) - F(3) = \lim_{M\to+\infty} M - 5\ln|M+4| - 3 + 5\ln 7 = +\infty$$

DIVERGE

- $\int_{-\infty}^{0} x^2 e^x\,dx$ $D_f: \mathbb{R}$

$$= \lim_{M\to+\infty} \int_{-M}^{0} x^2 e^x\,dx$$

$$F(x) = \int x^2 \underbrace{e^x}_{g'(x)}\,dx = x^2 e^x - \int 2x \underbrace{e^x}_{g'(x)}\,dx =$$

$$F(x) = \int x^2 \underbrace{e^x}_{g'(x)} dx = x^2 e^x - \int 2x \underbrace{e^x}_{g'(x)} dx =$$

$$= x^2 e^x - 2\left[x e^x - \int e^x dx\right] =$$

$$= x^2 e^x - 2x e^x + 2e^x + c$$

$$\lim_{M \to +\infty} F(0) - F(-M) = \lim_{M \to +\infty} 2e^0 - \left[M^2 e^{-M} + 2Me^{-M} + 2e^{-M}\right] =$$

$$= \lim_{M \to +\infty} 2 - M^2 e^{-M} - 2Me^{-M} - 2e^{-M} = 2$$

CONVERGE

CRITERI DI CONVERGENZA

- **CONFRONTO**: siano f e $g \geq 0$ definite in $[a;b)$
  e $f(x) \leq g(x) \quad \forall x \in [a;b)$
  $\Rightarrow \int_a^b f(x)dx \leq \int_a^b g(x)\, dx$

- **CONFRONTO ASINTOTICO**: siano f e $g \geq 0$ definite in $[a;b)$
  e $f(x) \sim g(x)$ per $x \to b$

  allora $\int_a^b f(x) dx$ e $\int_a^b g(x) dx$ o CONVERGONO o DIVERGONO

- **CONVERGENZA ASSOLUTA**:
  $\int_a^b |f(x)|\, dx$

  se gli integrali convergono convergono anche semplicemente (senza modulo)