11/12/2020

venerdì 11 dicembre 2020 08:30

SERIE NUMERICHE

$a_1, a_2 \dots \quad \sum_{n=1}^{\infty} a_n$

$S_k \to$ somma dei primi k termini

$$S_k = \sum_{n=1}^{k} a_n$$

$\lim_{k \to \infty} S_k =$
- $\to \exists$ finito la serie converge
- $\to = \pm\infty$ la serie diverge
- $\to \nexists$ la serie e' irregolare o oscillante

es. $\sum_{n=0}^{\infty} q^n = q^0 + q^1 + q^2 + \dots$

$$S_k = \sum_{n=0}^{k} q^n = q^0 + q + q^2 + \dots + q^k = \begin{cases} k+1 & \text{se } q = 1 \\ \frac{1-q^{k+1}}{1-q} & \text{se } q \neq 1 \end{cases}$$

$$(1+q+q^2+\dots+q^k)(1-q) = 1-q^{k+1}$$

$\lim_{k \to +\infty} S_k =$
- $\to +\infty$ se $q \geq 1$
- $\to \frac{1}{1-q}$ se $-1 < q < 1$
- $\to \nexists$ se $q \leq -1$

$\Rightarrow$ la serie geometrica di ragione q

DIVERE se $q \geq 1$
CONVERGE $-1 < q < 1$
IRREGOLARE se $q \leq -1$

- $\sum_{n=1}^{\infty} \frac{1}{n^2+n} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} \dots \quad \Big| \ \frac{1}{n^2+n} = \frac{1}{n} - \frac{1}{n+1}$

$$S_k = \sum_{n=1}^{k} \frac{1}{n^2+n} = \sum_{n=1}^{k} \left(\frac{1}{n} + \frac{1}{n+1} \right) =$$

$$= \left(1 - \cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \cancel{\dots} \dots + \left(\frac{1}{k} - \frac{1}{k+1}\right) =$$

$$= 1 - \frac{1}{k+1} \xrightarrow{k \to +\infty} 1$$

Si chiama serie di MENGOLI e' il piu' immediato esempio delle serie TELESCOPICHE

TEOREMA (LA CONDIZIONE NECESSARIA)

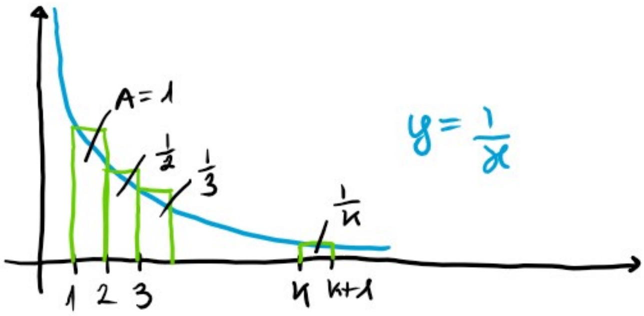
$\sum_{n=1}^{+\infty} a_n$ CONVERGE $\Rightarrow a_n \to 0$

NON VALE IL CONTRARIO

es.

$\sum^{\infty} \frac{1}{\ } = 1 + \frac{1}{\ } + \frac{1}{\ } + \dots$ DIVERGE es.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots \quad \text{DIVERGE}$$



$$S_k = \sum_{n=1}^{k} \frac{1}{n} = 1 + \frac{1}{2} + \dots + \frac{1}{k} \geq \int_1^{k+1} \frac{1}{x}\, dx$$

$$S_k \geq \ln(k+1) \xrightarrow{k \to +\infty} +\infty$$

la serie diverge

$$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} \begin{cases} \to \text{CONVERGE se } \alpha > 1 \\ \to \text{DIVERGE se } \alpha \leq 1 \end{cases}$$

DIM (del teorema)

$\exists$ finito il $\underbrace{\lim_{k \to \infty} S_k = S}_{\text{serie converge}}$ $\quad \lim_{k \to \infty} S_{k+1} = S$

$$\lim_{k \to \infty} (S_{k+1} - S_k) = 0$$

$$\sum_{n=1}^{k+1} a_n - \sum_{n=1}^{k} a_n = a_{k+1}$$

$$\underbrace{\lim_{k \to \infty} a_{k+1} = 0}_{a_n \to 0}$$

Se $a_n \not\to 0 \Rightarrow \sum a_n$ NON CONVERGE

SERIE A TERMINI POSITIVI ($a_n \geq 0$)

Proprietà:

- Una $\sum a_n$ con $a_n \geq 0$ non può essere irregolare

($S_{k+1} = S_k + a_{k+1} \geq S_k \Rightarrow S_k$ è MONOTONA)
[teorema di monotonia]

$$\sum \underbrace{(\cos n)^2}_{a_n} \quad \text{DIVERGE}$$

$a_n \geq 0$ la serie è a termini positivi

$a_n \not\to 0 \Rightarrow \sum a_n$ NON CONVERGE

$a_n \geq 0$ la serie è a termini positivi

$a_n \not\to 0 \Rightarrow \sum a_n$ NON CONVERGE

4 CRITERI:

① CRITERIO del confronto

$\sum_{n=1}^{\infty} a_n$ e $\sum_{n=1}^{\infty} b_n$ $\quad a_n, b_n \geq 0 \quad \forall n \quad a_n \leq b_n$

$\sum b_n$ CONVERGE $\Rightarrow \sum a_n$ CONVERGE

$\sum a_n$ DIVERGE $\Rightarrow \sum b_n$ DIVERGE

DIM

$A_k = \sum_{n=1}^{k} a_n \qquad B_k = \sum_{n=1}^{k} b_n \qquad A_k \leq B_k$

$\lim A_k \leq \lim B_k$

finito $\Leftarrow$ se finito

se ∞ $\Rightarrow$ inf

OSS. il criterio è valido anche se $a_n \leq b_n$ anche solo definitivamente

es. $\sum_{n=1}^{\infty} \frac{2+\cos n}{3n^2}$

$\frac{2+\cos n}{3n^2} \leq \frac{3}{3n^2} = \frac{1}{n^2}$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ CONVERGE

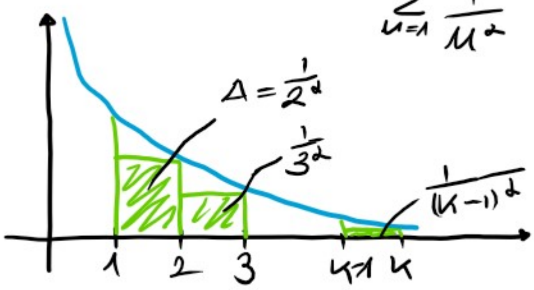
$\Rightarrow$ se CONVERGE anche $\sum \frac{2+\cos n}{3n^2}$ CONVERGE

$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \dots = \frac{\pi^2}{6}$

$\sum \frac{1}{n}$ diverge $\quad \alpha < 1 \quad \frac{1}{n^\alpha} > \frac{1}{n} \quad \sum \frac{1}{n^\alpha}$ DIVERGE

se $\alpha > 1$

$\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$



$S_k = \sum_{n=1}^{k} \frac{1}{k^\alpha} \leq 1 + \int_1^k \frac{1}{x^\alpha} dx$

$S_k \leq 1 + \left[\frac{1}{[illegible]} \frac{1}{x^{\alpha-1}} \right]_1^k = 1 + \frac{1}{[illegible]} \left(1 - \frac{1}{[illegible]} \right) \xrightarrow{k \to +\infty} 1 + \frac{1}{[illegible]}$

$$S_k = \sum_{n=1} \frac{1}{k^\alpha} \leq 1 + \int_1 \frac{1}{x^\alpha} dx$$

$$S_k \leq 1 + \left[\frac{1}{1-\alpha} \frac{1}{x^{\alpha-1}} \right]_1^k = 1 + \frac{1}{\alpha-1}\left(1 - \frac{1}{k^\alpha}\right) \xrightarrow{k \to +\infty} 1 + \frac{1}{\alpha-1}$$

$\Rightarrow$ CONVERGE

$\frac{1}{n^\alpha}$ se $\alpha \leq 1$ DIVERGE

se $\alpha > 1$ CONVERGE

② criterio del confronto asintotico

$\sum a_n \quad \sum b_n \quad a_n, b_n \geq 0 \quad a_n \sim b_n$

$\Rightarrow \sum a_n$ e $\sum b_n$ hanno lo stesso carattere

DIM

$\lim_{n \to \infty} \frac{a_n}{b_n} = 1 \quad \forall \varepsilon > 0 \; \exists \bar{n} : \forall n \geq \bar{n} \; \left| \frac{a_n}{b_n} - 1 \right| < \varepsilon$

$\varepsilon = \frac{1}{2} \quad \exists \bar{n} : \left| \frac{a_n}{b_n} - 1 \right| < \frac{1}{2} \Rightarrow \quad \frac{1}{2} b_n < a_n < \frac{3}{2} b_n$

$\forall n \geq \bar{n}$ (definitivamente)

es. $\sum \frac{n^2 + \ln n}{n^4 - e^{-n}}$

$$\frac{n^2 + \ln n}{n^4 - e^{-n}} \sim \frac{n^2}{n^4} = \frac{1}{n^2} \quad \text{CONVERGE}$$

$$\sum \left(\frac{1}{\sqrt{n}} - \sin \frac{1}{\sqrt{n}} \right)$$

$$t \to 0 \quad t - \sin t = \cancel{t} - \left(\cancel{t} - \frac{t^3}{3!} + \dots \right) =$$

$$= \frac{t^3}{3!} + o(t^3)$$

$$t - \sin t \sim \frac{t^3}{3!}$$

$$n \to +\infty \quad \left(\frac{1}{\sqrt{n}} - \sin \frac{1}{\sqrt{n}} \right) \sim \left(\frac{\left(\frac{1}{\sqrt{n}}\right)^3}{6} \right) =$$

$$= \frac{1}{6 n^{3/2}} \quad \alpha > 1 \text{ CONVERGE}$$

$$\sum \frac{1}{\sh n} = \sum \frac{2}{e^n - e^{-n}}$$

$$\frac{2}{e^n - e^{-n}} \sim \frac{2}{e^n} \qquad \sum \frac{2}{e^n} = 2 \sum \left(\frac{1}{e}\right)^n$$

come serie geometrica di ragione q

$-1 < \frac{1}{e} < 1 \Rightarrow$ la serie converge

③ CRITERIO del rapporto

$\sum a_n \qquad a_n > 0$

$\lim_{n \to \infty} \frac{a_{n+1}}{a_n}$
- $\rightarrow \nexists$ NON si applica il criterio
- $\rightarrow > 1$ DIVERGE
- $\rightarrow = 1$ non sappiamo dire niente
- $\rightarrow \in [0,1)$ CONVERGE

se $\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = 1^+ \Rightarrow \sum$ DIVERGE

DIM

$\lim \frac{a_{n+1}}{a_n} = \ell < 1 \Rightarrow \sum$ CONVERGE

$\forall \varepsilon > 0 \ \exists \bar{n} : \forall n > \bar{n} \ \left| \frac{a_{n+1}}{a_n} - \ell \right| < \varepsilon$

$(\ell - \varepsilon) a_n < a_{n+1} < (\ell + \varepsilon) a_n$ scelgo $\varepsilon : \ell + \varepsilon = \gamma < 1$

$\forall n \geq \bar{n}$ (definitivamente) $a_{n+1} < \gamma a_n$

$a_{\bar{n}+1} \leq \gamma a_{\bar{n}} \qquad a_{\bar{n}+2} < \gamma a_{\bar{n}+1} \leq \gamma^2 a_{\bar{n}}$

$a_{\bar{n}+n} \leq \gamma^n a_{\bar{n}}$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\bar{n}-1} a_n + \underbrace{\sum_{n=\bar{n}}^{+\infty} a_n}_{\sum_{n=0}^{\infty} a_{\bar{n}+n}} \qquad \sum_{n=0}^{\infty} \gamma^n a_{\bar{n}} \text{ CONVERGE}$$

$\sum \frac{n!}{n^n}$

$$\lim \frac{a_{n+1}}{a_n} = \lim \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} =$$

$$= \lim \frac{\cancel{(n+1)}!}{(n+1)^{\cancel{n+1}}} \cdot \frac{n^n}{\cancel{n!}} =$$

$$= \lim \left(\frac{n}{n+1}\right)^n = \frac{1}{e} \Rightarrow \text{CONVERGE}$$