11/12/2020

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INTEGRALI IMPROPI

- $\int_0^1 \frac{x}{x^2-1}\,dx$ $\qquad f(x)$ continua $x \neq \pm 1$

$f: [0;1) \to \mathbb{R}$ continua

NON VALE il teorema fondamentale del calcolo integrale

$$\lim_{\varepsilon \to 0} \int_0^{1-\varepsilon} \frac{x}{x^2-1}\,dx$$

$$\int \frac{x}{x^2-1}\,dx = \frac{1}{2}\int \frac{2x}{x^2-1}\,dx = \frac{1}{2}\ln|x^2-1|+c$$

$$\int_0^{1-\varepsilon} \frac{x}{x^2-1}\,dx = \frac{1}{2}\ln|\varepsilon^2-2\varepsilon|$$

$$\lim_{\varepsilon \to 0} \frac{1}{2}\ln|\varepsilon^2-2\varepsilon| = -\infty \quad \text{diverge}$$

- $\int_0^{+\infty} \frac{\ln^3(x)}{x}\,dx$ $\qquad f(x)$ continua $x > 0$

$$\int_0^1 \frac{\ln^3(x)}{x}\,dx + \int_1^{+\infty} \frac{\ln^3(x)}{x}\,dx =$$

$$= \lim_{\varepsilon \to 0} \int_{0+\varepsilon}^1 \frac{\ln^3(x)}{x}\,dx + \lim_{R \to +\infty} \int_1^R \frac{\ln^3(x)}{x}\,dx =$$

$$\int \frac{\ln^3(x)}{x}\,dx = \frac{\ln^4(x)}{4} + c$$

$$= \lim_{\varepsilon \to 0} \left[\frac{\ln^4(x)}{4}\right]_{0+\varepsilon}^1 + \lim_{R \to +\infty} \left[\frac{\ln^4(x)}{4}\right]_1^R$$

$$\lim_{\varepsilon \to 0} -\frac{\ln^4(\varepsilon)}{4} + \lim_{R \to +\infty} \frac{\ln^4 M}{4} = -\infty + \infty$$

in ogni caso entrambi divergono

$\Rightarrow$ l'integrale diverge

- $\int_1^{+\infty} \frac{\operatorname{arctg} x}{x^2+4}\,dx =$

$$= \lim_{R \to +\infty} \int_1^R \frac{\operatorname{arctg} x}{x^2+4}\,dx$$

$$\int \frac{\operatorname{arctg} x}{x^2+4}\,dx =$$

CRITERIO del confronto o confronto asintotico

$\forall k > 1$ grande

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$\int_1^k f(x)dx$ CONVERGE il problema lo ho all'infinito

per $x \to +\infty$ $f(x) \sim \frac{\pi/2}{x^2} = \frac{c}{x^2}$

$\int_k^{+\infty} f(x)\,dx$ ha lo stesso carattere di $\int_k^{+\infty} \frac{c}{x^2}\,dx$

$$c \lim_{M \to +\infty} \int_k^M \frac{1}{x^2}\,dx = c \lim_{M \to +\infty} \left[-\frac{1}{x} \right]_k^M =$$

$$c \lim_{M \to +\infty} -\frac{1}{M} + \frac{1}{k} = \frac{c}{k}$$

$\Rightarrow$ siccome converge allora anche $\int f(x)$ converge

- $\displaystyle\int_1^{+\infty} \frac{e^x}{e^{2x}-1}\,dx = \lim_{M \to +\infty} \int_1^M \frac{e^x}{e^{2x}-1}\,dx$

$$\int \frac{e^x}{e^{2x}-1}\,dx$$

$e^x = t \qquad x = \ln t$

$$\int \frac{\cancel{t}}{t^2-1} \cdot \frac{1}{\cancel{t}}\,dt = \int \frac{1/2}{t-1}\,dt + \int -\frac{1/2}{t+1}\,dt =$$

$$= \frac{1}{2} \ln|t-1| - \frac{1}{2} \ln|t+1| + c$$

$$= \frac{1}{2} \ln|e^x-1| - \frac{1}{2} \ln|e^x+1| + c$$

$$\int_1^M f(x)\,dx = F(M) - F(1) = \frac{1}{2} \ln(e^M-1) - \frac{1}{2} \ln(e^M+1) -$$

$$- \frac{1}{2} \ln(e-1) + \frac{1}{2} \ln(e+1)$$

$$\lim_{M \to +\infty} \frac{1}{2} \ln\left(\frac{e^M-1}{e^M+1}\right) - \frac{1}{2} \ln\left(\frac{e-1}{e+1}\right)$$

$$= \lim_{M \to +\infty} \frac{1}{2} \ln\left[\frac{e^M(1-e^{-M})}{e^M(1+e^{-M})}\right] - \frac{1}{2} \ln\left(\frac{e-1}{e+1}\right)$$

$$= -\frac{1}{2} \ln\left(\frac{e-1}{e+1}\right) \quad \text{CONVERGE}$$

- $\displaystyle\int_1^{+\infty} \frac{e^{x^2}}{1+e^{(2x^2)}}$

per $k > 1$ grande

$\int^k f(x)\,dx$ CONVERGE

per $k>1$ grande

$\int_1^k f(x)\,dx$ CONVERGE

$F(k)-F(1)$

$f(x)>0$ per $x\in[1;+\infty)$

per $x\to+\infty$ $\quad f(x)\sim\frac{e^{x^2}}{e^{2x^2}}=e^{-x^2}$

$\int_k^{+\infty}f(x)\,dx$ ha lo stesso carattere di

$\int_k^{+\infty}e^{-x^2}\,dx$

$e^{-x^2}\le e^{-x}$) CONFRONTO

$\int_k^{+\infty}e^{-x^2}\,dx\le\int_k^{+\infty}e^{-x}\,dx$

$\int_k^{+\infty}e^{-x}\,dx=\lim_{M\to+\infty}\int_k^{M}e^{-x}\,dx=\lim_{M\to+\infty}\left[-e^{-x}\right]_k^M=$

$=\lim_{M\to+\infty}-e^{-M}+e^{-k}=e^{-k}$ CONVERGE

$\Rightarrow$ anche l'integrale di e^{-x^2} converge e converge anche l'integrale di $f(x)$

- $f(x)=\frac{x^2-2x}{(x-1)^{4/3}(x-2)^{10/7}}$

$$f(x)=\begin{cases}\frac{x}{(x-1)^{4/3}(x-2)^{3/7}} & x>2\\[1ex] \frac{-x}{(x-1)^{4/3}(x-2)^{3/7}} & x<2\ \ x\neq1\end{cases}$$

- $\int_3^5 f(x)\,dx$ CONVERGE SICURO
- $\int_3^{+\infty}f(x)\,dx$

per $x\to+\infty$ $\quad f(x)\sim\frac{x}{x^{4/3}x^{3/7}}=\frac{1}{x^{16/21}}$

$\int_3^{+\infty}f(x)\,dx$ ha lo stesso carattere di $\int_3^{+\infty}\frac{1}{x^{16/21}}\,dx$

$\alpha<1$ l'integrale diverge

- $\int_2^3 f(x)\,dx$

per $x\to2^+$ $\quad f(x)\sim\frac{2}{(2-1)^{4/3}(x-2)^{3/7}}=\frac{2}{(x-2)^{3/7}}$

$\int_2^3 f(x)\,dx$ ha lo stesso carattere $\int_2^3\frac{2}{(x-2)^{3/7}}\,dx$

J_2 $\qquad\qquad$ $J_2\,(x-2)$

$\alpha<1$ CONVERGE

$$-\int_0^1 f(x)\,dx$$

per $x\to 1^-$ $\quad f(x)\sim -\frac{1}{(x-1)^{4/3}(1-2)^{3/7}} \sim -\frac{1}{(x-1)^{4/3}}$

$\int_0^1 -f(x)\,dx$ ha lo stesso carattere di $\int_0^1 \frac{1}{(x-1)^{4/3}}$

$\alpha>1$ Diverge

- $\int_0^1 \frac{\sin x \ln(x)}{x^{9/4}}\,dx$

$\sin x>0$ per $x\in(0;1)$

$\ln(x)<0$ per $x\in(0;1)$ $\qquad f(x)<0$ per $x\in(0;1)$

$\lim_{x\to 0} f(x)=\infty$

per $x\to 0^+$ $\quad |f(x)| = \left|\frac{\sin x}{x}\frac{\ln x}{x^{5/4}}\right| \sim \left|\frac{\ln x}{x^{5/4}}\right|$ $\quad g(x)$

per $x\in(0;e^{-1}]$ $\qquad |\ln(x)|\geq 1$

$-\frac{\ln(x)}{x^{5/4}} = \frac{|\ln(x)|}{x^{5/4}} \geq \underbrace{\frac{1}{x^{5/4}}}_{h(x)}$ $\qquad x\in(0;1/e]$

$\int_0^1 g(x)\,dx \geq \int_0^1 h(x)\,dx$

$\alpha>1$ DIVERGE

- $F(x)=\int_0^x \frac{te^t}{\sqrt{1-t}}\,dt$ $\qquad f(t)=\frac{te^t}{\sqrt{1-t}}$

$f(t)$ è definita per $1-t>0$

$t<1$

$D_F=(-\infty;1)$

$\lim_{x\to 1^-} F(x) = \lim_{x\to 1^-}\int_0^x \frac{te^t}{\sqrt{1-t}}\,dt$

per $t\to 1$ $\quad f(t)\sim \frac{e}{(1-t)^{1/2}} = g(t)$

$\int_0^1 g(t)\,dt$

$\alpha<1$ CONVERGE

$\int_0^1 g(t)\,dt$ CONVERGE $\Rightarrow \lim_{x\to 1^-} F(x) = c$

$D_F = (-\infty; 1]$ con $F(1)=c$

$\lim_{x\to -\infty} F(x) = \lim_{x\to-\infty} \int_0^x g(t)dt = -\int_{-\infty}^0 g(t)dt$

per $t\to -\infty$ $g(t)<0$

$-g(t) \sim \frac{(-t)e^t}{(-t)^{1/2}} = (-t)^{1/2}e^t = \sqrt{-t}\,e^t\,dt$

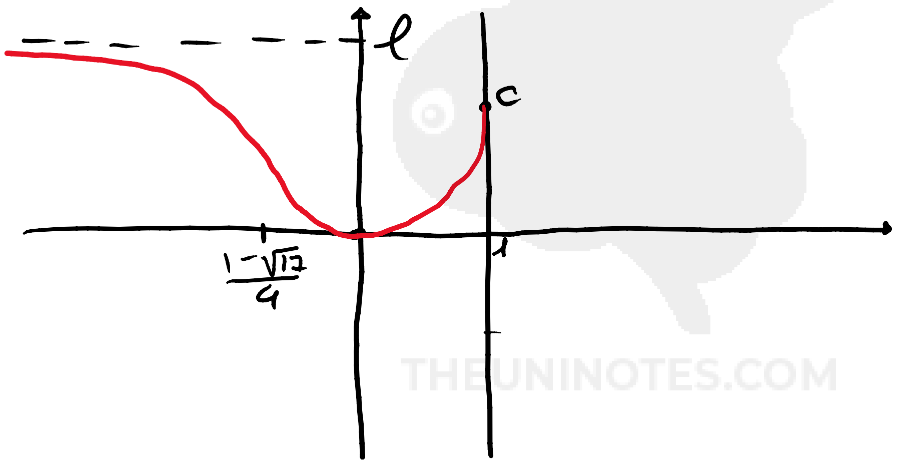
$h(t) < \frac{1}{t^2}$

$\int_{-\infty}^0 h(t)dt \leq \int_{-\infty}^0 \frac{1}{t^2}dt$ CONVERGE

$\int_{-\infty}^0 -g(t)dt$ CONVERGE

$\lim_{x\to-\infty} F(x) = \ell$ CONVERGE

$F(x)$ è derivabile in $(-\infty;1) \Rightarrow$ continua in $(-\infty;1]$



$F'(x) = g(x) = \frac{xe^x}{\sqrt{1-x}}$ $\quad F'(x)=0 \quad x=0$

$F'(x)>0 \quad e^x>0$
$\sqrt{1-x}>0$

F cresce per $x>0$ $\quad x=0$ min

$F(0)=0$

$\lim_{x\to1^-} F'(x) = \lim_{x\to1^-} \frac{xe^x}{\sqrt{1-x}} = +\infty$ tg verticale

$F''(x) = \frac{e^x(x+1)\sqrt{1-x} + xe^x\frac{1}{2\sqrt{1-x}}}{(1-x)} =$

$= \frac{e^x[(x+1)(1-x)+x/2]}{(1-x)\sqrt{1-x}} = \frac{e^x}{(1-x)^{3/2}}\left[1-x^2+\frac{x}{2}\right]$

$F''(x)=0 \quad 1-x^2+\frac{x}{2}=0$

$= \frac{\ldots}{(1-x)\sqrt{1-x}} = \frac{\ldots}{(1-x)^{3/2}}\left[\ldots \frac{x}{2}\right]$

$F''(x)=0 \qquad 1-x^2+\frac{x}{2}=0$

$-2x^2+x+2=0$

$\Delta = 1+16=17$

$x=\frac{-1\pm\sqrt{17}}{-4} \to \frac{1+\sqrt{17}}{4} > 1$ fuori dal dominio

$\to \frac{1-\sqrt{17}}{4}$

$F''(x)>0 \qquad e^x>0$

$(1-x)^{3/2}>0$

$-2x^2+x+2>0$

$\frac{1-\sqrt{17}}{4}<x<\frac{1+\sqrt{17}}{4} \longrightarrow$ concavità verso l'alto

FLESSO $x=\frac{1-\sqrt{17}}{4}$

COMPITO $\qquad F(x)=\int_0^x \ln\left(1+\frac{t^2}{t^2+1}\right)dt$