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TEOREMA del calcolo d'integrale

$f \in \mathcal{C}[a,b]$, F primitiva

$$\int_a^b f(x)\,dx = F(b)-F(a) = [F(x)]_a^b$$

DIM

$$F(b)-F(a) = F(x_n)-F(x_0)$$
$$= F(x_n)-F(x_{n-1})+F(x_{n-1})-\dots-F(x_1)+F(x_1)-F(x_0)$$
$$= \sum_{j=1}^{n} F(x_j)-F(x_{j-1})$$

Se g è continua in $[c,d]$ e anche derivabile

x TEO LAGRANGE $\exists\, t \in (c,d) : g'(t) = \dfrac{g(d)-g(c)}{d-c}$

$$\Rightarrow g(d)-g(c) = g'(t)\cdot(d-c)$$

$$= \sum_{j=1}^{n} F(x_j)-F(x_{j-1}) = \sum_{j=1}^{n} f(t_j)\cdot(x_j-x_{j-1})$$

per un opportuno t_j

$$F(b)-F(a) = \sum_{j=1}^{n} f(t_j)(x_j-x_{j-1}) \xrightarrow{n\to\infty} \int_a^b f(x)\,dx$$

INTEGRALE INDEFINITO:

$$\int f(x)\,dx = F(x)+C$$

es. $\int \cos x\,dx = \sin x + c$

Tutte le funzioni continue sono integrabili

$\Rightarrow \int_a^b f(x)\,dx$ è un numero e ha significato

Tutte le funzioni continue hanno la primitiva

$$\int_a^b f(x)\,dx = F(b)-F(a)$$

es. $f(x) = e^{-x^2} \rightarrow$ non sappiamo scrivere la primitiva essa esiste ma non sappiamo scriverla

METODO d'integrazione:

INTEGRALI IMMEDIATI $\rightarrow$ sono le derivate lette al contrario

INTEGRALI IMMEDIATI → sono le derivate lette al contrario

es. $\int \left(x^3+\sqrt{x}+\frac{1}{x}\right)dx = \frac{1}{4}x^4+\frac{2}{3}x^{\frac{3}{2}}+\ln|x|+c$

es. $\int \left(e^{2x}+\sin 3x-\frac{1}{1+x^2}\right)dx = \frac{1}{2}e^{2x}-\frac{1}{3}\cos 3x-\operatorname{arctg} x+c$

INTEGRAZIONE PER PARTI

F e G primitive di f e g su [a, b]

$$\int_a^b F(x)g(x)dx=\left[F(x)G(x)\right]_a^b-\int_a^b f(x)G(x)dx$$

$$\int F(x)g(x)dx = F(x)G(x)-\int f(x)G(x)dx$$

DIM

$$\left[F(x)G(x)\right]'=f(x)G(x)+F(x)g(x)$$

$$\int_a^b \left(f(x)G(x)+F(x)g(x)\right)dx=\left[F(x)G(x)\right]_a^b$$

$$\int_a^b F(x)g(x)dx=\left[F(x)G(x)\right]_a^b-\int_a^b f(x)G(x)dx$$

es. $\int \underbrace{x}_{F}\underbrace{\cos x}_{g}\,dx = x\sin x-\int 1\cdot\sin x\,dx =$

$= x\sin x+\cos x+c$

COMPITO

$\int \underbrace{x^2}_{F}\underbrace{e^{2x}}_{g}dx = x^2\cdot\frac{1}{2}e^{2x}-\int \underbrace{\cancel{2}x}_{f}\frac{1}{\cancel{2}}\underbrace{e^{2x}}_{g}dx=\frac{1}{2}x^2e^{2x}-\left(\frac{x}{2}e^{2x}-\frac{1}{2}\int e^{2x}dx\right)=$

↳ si applica l'integrale per parti due volte

$=\frac{1}{2}x^2e^{2x}-\frac{1}{2}xe^{2x}+\frac{1}{4}e^{2x}+c$

$\int \ln x\,dx=\int \underbrace{1}_{g}\cdot\underbrace{\ln x}_{F}\,dx = x\ln x-\int x\frac{1}{x}dx = x\ln x-x+c$

COMPITO

$\int e^{2x}\sin 3x\,dx = \dots + k\int e^{2x}\sin 3x\,dx$

INTEGRAZIONE PER SOSTITUZIONE

$f\in\mathscr{C}(\alpha,\beta)$ φ invertibile e derivabile in $[a,b]$
φ' integrabile in $[a,b]$

$\tau=\varphi(t)$ $\alpha=\varphi(a)$ $\beta=\varphi(b)$

$$\int_a^b f(\varphi(t))\varphi'(t)dt=\int_\alpha^\beta f(\tau)d\tau$$

$\int f(\varphi(t))\varphi'(t)dt=\int f(\tau)d\tau$ con $\tau=\varphi(t)$

DIM

$\int f(\varphi(t))\varphi'(t)dt = \int f(\tau)d\tau$ con $\tau = \varphi(t)$

DIM

$\int_\alpha^\beta f(\tau)d\tau$ f continua quindi ha primitiva

$$\int_\alpha^\beta f(\tau)d\tau = [F]_\alpha^\beta = F(\beta) - F(\alpha) = F(\varphi(b)) - F(\varphi(a))$$

$$\frac{d}{dt}[F(\varphi(t))] = f(\varphi(t))\varphi'(t)$$

$$\int_a^b f(\varphi(t))\varphi'(t)dt = [F(\varphi(t))]_a^b = F(\varphi(b)) - F(\varphi(a))$$

es. $\int \frac{e^x}{\cos^2(e^x)}dx = \int \frac{1}{\cos^2(e^x)} \underbrace{e^x dx}_{du}$

$e^x = u \quad e^x dx = 1 \cdot du$

$$\int \frac{1}{\cos^2 u}du = tg\, u + c = tg(e^x) + c$$

es. $\int \frac{e^{2x}+x}{e^{2x}+x^2}dx = \frac{1}{2}\int \frac{2e^{2x}+2x}{e^{2x}+x^2}dx = \frac{1}{2}\ln\underbrace{|e^{2x}+x^2|}+c$

il valore ass. non serve perché sempre positiva

$$\int \frac{g'(x)}{g(x)}dx = \ln|g(x)| + c$$

$g(x) = u \quad g'(x)dx = du$

es. $\int \underbrace{(\sin^3 x - \sin x)}_{f(\sin x)}\cos x\, dx = \int (u^3 - u)du =$

$f(t) = t^3 - t \qquad \cos x\, dx = du$

$$= \frac{1}{4}u^4 - \frac{1}{2}u^2 + c = \frac{1}{4}\sin^4 x - \frac{1}{2}\sin^2 x + c$$

es. $\int e^{\sqrt{x}}dx =$

$x = t^2$

$dx = 2t\,dt$

$$\int e^t \cdot 2t\,dt = 2\int t e^t dt = 2\left(te^t - \int e^t dt\right) =$$

$$= 2(te^t - e^t) + c = 2e^{\sqrt{x}}(\sqrt{x} - 1) + c$$

es. $\int x\sqrt{1-x^2}\,dx = \frac{1}{2}\int \sqrt{1-x^2}\,2x\,dx$

$1 - x^2 = t$

$-2x\,dx = dt$

$$-\frac{1}{2}\int \sqrt{t}\cdot dt = -\frac{1}{3}t^{3/2} + c = -\frac{1}{3}(1-x^2)^{\frac{3}{2}} + c$$

es. $\int \sqrt{1-x^2}\,dx =$

$$x = \sin t \qquad dx = \cos t\,dt$$

$$\int \underbrace{\sqrt{1-\sin^2 t}}_{\cos t}\,\cos t\,dt = \int \cos^2 t\,dt$$

$$\cos^2 t = \frac{1+\cos 2t}{2}$$

$$\frac{1}{2}\int (1+\cos 2t)\,dt = \frac{1}{2}\left(t + \frac{1}{2}\sin 2t\right) + c =$$

$$= \frac{1}{2}\left(t + \sin t \cos t\right) + c = \frac{1}{2}\left(\arcsin x + x\sqrt{1-x^2}\right) + c$$

es. $\int x|x|\,dx = \left\{\begin{array}{lll} \int x^2\,dx & x>0 & \rightarrow \frac{1}{3}x^3 + c \\ \int -x^2\,dx & x<0 & \rightarrow -\frac{1}{3}x^3 + c \end{array}\right\} = \frac{1}{3}x^2|x| + c$

COMPITO

$\int x|x-2|\,dx \qquad \int_0^{2\pi} \lfloor x \rfloor\,dx$ ☐

es. $\int \sqrt{1+x^2}\,dx = \int \underbrace{\sqrt{1+\mathrm{sh}^2 t}}_{\mathrm{ch}\,t} \cdot \mathrm{ch}\,t\,dt = \int \mathrm{ch}^2 t\,dt =$

$x = \mathrm{sh}\,t$

$dx = \mathrm{ch}\,t\,dt$

$\hookrightarrow \frac{1}{2}(1 + \mathrm{ch}\,2t)$

$$= \frac{1}{2}\int (1+\mathrm{ch}\,2t)\,dt = \frac{1}{2}\left(1 + \frac{1}{2}\mathrm{sh}\,2t\right) + c$$

$$\mathrm{sh}\,2t = 2\,\mathrm{sh}\,t\,\mathrm{ch}\,t$$

$$= \frac{1}{2}\left(t + \mathrm{sh}\,t\,\mathrm{ch}\,t\right) + c$$

$$x = \mathrm{sh}\,t = \frac{e^t - e^{-t}}{2} \Rightarrow e^t - 2x - e^{-t} = 0$$

$$e^{t^2} - 2xe^t - 1 = 0$$

$$e^t = x \pm \sqrt{x^2+1}$$

$\hookrightarrow$ scarto il meno

$$t = \ln\left(x + \sqrt{x^2+1}\right)$$

$$\frac{1}{2}\left(\ln\left(x+\sqrt{x^2+1}\right) + x\left(1+\sqrt{1+x^2}\right)\right) + c$$