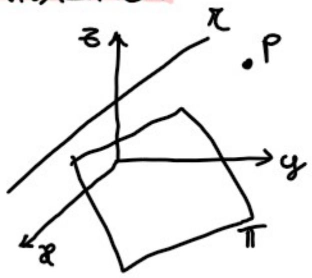
20/11/2020

venerdì 13 novembre 2020 14:33

GEOMETRIA ANALITICA

In $\mathbb{R}^3$



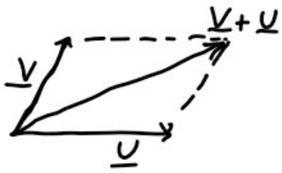
PUNTO P

RETTA r

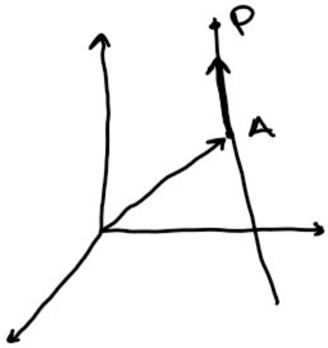
PIANO π

vettori $\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbb{R}^3$

$$\underline{v} + \underline{u} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} v_1+u_1 \\ v_2+u_2 \\ v_3+u_3 \end{bmatrix}$$



$$\lambda \underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \lambda = \begin{bmatrix} \lambda v_1 \\ \lambda v_2 \\ \lambda v_3 \end{bmatrix} \qquad \lambda \neq 0$$



$A = (a_1, a_2, a_3) \qquad P = (x, y, z)$

$O = (0,0,0)$

$$\overrightarrow{OA} = (a_1 - 0, a_2 - 0, a_3 - 0) = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$\overrightarrow{OP} = \overrightarrow{OA} + \tau \underline{v}$$

$$\overrightarrow{OP} = (x-0, y-0, z,0) = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

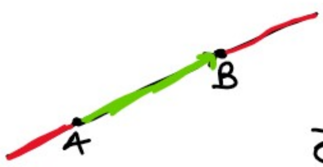
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \tau \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \rightarrow \begin{cases} x = a_1 + \tau v_1 \\ y = a_2 + \tau v_2 \\ z = a_3 + \tau v_3 \end{cases}$$

es. $A(1,2,0) \qquad \underline{v} = \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix}$

$$OP = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \tau \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix}$$

$$\begin{cases} x = 1 - \tau \\ y = 2 + 3\tau \\ z = 5\tau \end{cases}$$

es. $A = (1,-3,2) \qquad B = (4,0,7)$



$$\overrightarrow{AB} = \begin{bmatrix} 3 \\ 3 \\ 5 \end{bmatrix} = \underline{v}$$

$$\overrightarrow{OP} = \overrightarrow{OA} + \tau \overrightarrow{AB}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + \tau \begin{bmatrix} 3 \\ 3 \\ 5 \end{bmatrix} \rightarrow \begin{cases} x = 1 + 3\tau \\ y = -3 + 3\tau \\ z = 2 + 5\tau \end{cases}$$

es

$$\begin{cases} x = \tau + 7 & = 7 + \tau \\ y = -3 + 5\tau & = -3 + 5\tau \\ z = 2(1+\tau) & = 2 + 2\tau \end{cases} \qquad \underline{v} = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix}$$

se $\tau = 0$ $A = (7, -3, 2)$

se $\tau = 1$ $B = (8, 2, 4)$

es. $\underline{v} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$ $A = (1,1,1)$ determinare la retta

$$\vec{OP} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \tau \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \rightarrow \begin{cases} x = 1 + 2\tau \\ y = 1 - \tau \\ z = 1 + 3\tau \end{cases}$$

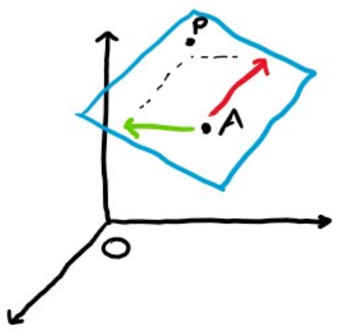
sia $\underline{w} = \begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix}$ $B = (2, 1/2, 5/2)$

$$\vec{OP} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1/2 \\ 5/2 \end{bmatrix} + s \begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix} \rightarrow \begin{cases} x = 2 - 4s \\ y = 1/2 + 2s \\ z = 5/2 - 6s \end{cases}$$

sono la stessa retta perché:

- $\underline{w} = -2\underline{v}$ => $\underline{v}$ e $\underline{w}$ hanno la stessa direzione
- in più le due rette passano per gli stessi punti
  posso sostituire B nella prima retta e trovare
  che appartiene alla retta e viceversa

$$\begin{cases} x = 1 + 2\tau = 2 \\ y = 1 - \tau = 1/2 \\ z = 1 + 3\tau = 5/2 \end{cases} \qquad \tau = \frac{1}{2}$$



$\underline{U}, \underline{V}$ vettori

$\vec{OP} = \vec{OA} + \tau \underline{V} + s \underline{U}$

$A = (a_1, a_2, a_3)$

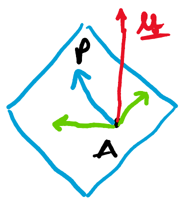
$P = (x, y, z)$

$\underline{V} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$ $\underline{U} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \tau \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} + s \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

$$\begin{cases} x = a_1 + \tau V_1 + s U_1 \\ y = a_2 + \tau V_2 + s U_2 \\ z = a_3 + \tau V_3 + s U_3 \end{cases}$$

PIANO IN FORMA PARAMETRICA



$\underline{v} \cdot \underline{u} = 0 \qquad v \perp u$

$\underline{u} \cdot \underline{u} = 0 \qquad u \perp u$

$\vec{AP} \perp \underline{u}$

se conosco $\underline{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad \vec{AP} = \begin{bmatrix} x & -a_1 \\ y & -a_2 \\ z & -a_3 \end{bmatrix}$

$$a(x - a_1) + b(y - a_2) + c(z - a_3) = 0$$

$$ax + by + cz - \underbrace{(aa_1 + ba_2 + ca_3)}_{d} = 0$$

PIANO IN FORMA CARTESIANA

es. $A = (1,0,1) \quad \underline{v} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \quad \underline{u} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$

$$\vec{OP} = \vec{OA} + t\underline{v} + s\underline{u}$$

$$OP = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} + s\begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$$

$$\begin{cases} x = 1 + 2t - s \\ y = t + s \\ z = 1 + t + 3s \end{cases}$$

se elimino i parametri

$s = y - t$

$$\begin{cases} x = 1 + 2t - y + t = 1 - y + 3t \\ z = 1 + t + 3y - 3t = 1 + 3y - 2t \end{cases}$$

$$t = \frac{z - 3y - 1}{-2} = \frac{1 + 3y - z}{2}$$

eq. cartesiana π: $x = 1 - y + \frac{3}{2}(1 + 3y - z)$

$$2x = 2 - 2y + 3 + 9y - 3z$$

$$2x - 7y + 3z - 5 = 0$$

$$\underline{u} = \begin{bmatrix} 2 \\ -7 \\ 3 \end{bmatrix}$$

$\underline{v} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} \quad \underline{u} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$

$\underline{u} \cdot \underline{v} = 2 \cdot 2 - 7 \cdot 1 + 3 = 0$

$\underline{u} \cdot \underline{u} = -2 - 7 + 2 = 0$

$$\underline{u} = \underline{v} \times \underline{u} = \begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ -1 & 1 & 3 \end{vmatrix} =$$

$$\underline{i}(3-1) - \underline{j}(6+1) + \underline{k}(2+1)$$

$$= 2i - 7j + 3k \qquad u = \begin{bmatrix} 2 \\ -7 \end{bmatrix}$$

$$\underline{i}(3-1)-\underline{j}(6+1)+\underline{k}(2+1)$$
$$=2\underline{i}-7\underline{j}+3\underline{k} \qquad u=\begin{bmatrix}2\\-7\\3\end{bmatrix}$$

$A=(1,0,1)$

$$\vec{PA}=\begin{bmatrix}x-1\\y\\z-1\end{bmatrix} \qquad \vec{PA}\cdot\underline{u}=0 \Rightarrow 2(x-1)-7y+3(z-1)=0$$
$$2x-7y+3z-5=0$$

l'equazione di un piano è sempre della stessa forma

$$ax+by+cz-d=0$$

quelle di u ↳ sostituisco il punto e trovo d

es

$A=(2;-1;0)$ $B=(0;4;-3)$ $C=(1;1;-1)$

$$\vec{AB}=\begin{bmatrix}-2\\5\\-3\end{bmatrix} \to \underline{u} \qquad \vec{AC}=\begin{bmatrix}-1\\2\\-1\end{bmatrix} \to \underline{v}$$

eq. parametrica

$$\vec{OP}=\vec{OA}+t\,\underline{v}+s\,\underline{u}$$

$$u=\underline{u}\times\underline{v}=\begin{vmatrix}i & j & k\\-2 & 5 & -3\\-1 & 2 & -1\end{vmatrix}=\underline{i}(-5+6)-\underline{j}(2-3)+\underline{k}(-4+5)$$

$$\underline{u}=\begin{bmatrix}1\\1\\1\end{bmatrix} \qquad u\cdot\underline{u}=0 \quad u\cdot\underline{v}=0$$

$$ax+by+cz+d=0$$
$$x+y+z+d=0$$
$$2-1+d=0 \qquad d=-1$$

$$ax+by+cz-1=0$$

$$ax+by+cz+d=0$$

$$\begin{cases}2a-b+d=0 & (A)\\4b-3c+d=0 & (B)\\a+b-c+d=0 & (C)\end{cases}$$

Sistema omogeneo di 3 eq. in 4 incognite es $P=(0,1,1)$

$$r=\begin{cases} x+y+z=0 \\ x-z-1=0 \end{cases}$$

↳ rappresentano 2 piani
⇒ la retta è vista come intersezioni tra due piani

metto il parametro

$$\begin{cases} x=1+t \\ y=-1-t-t=-1-2t \\ z=t \end{cases} \qquad \underline{v_r}=\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

scelgo un punto sulla retta $A=(1,-1,0)$

$$\vec{PA}=\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

$$\underline{u}=\begin{vmatrix} i & j & k \\ 1 & -2 & 1 \\ 1 & -2 & -1 \end{vmatrix}=\underline{i}(4)-\underline{j}(-1-1)+\underline{k}(0)$$

$$\underline{u}=\begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} \text{ "="} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$2x+y+d=0$$

$$1+d=0 \qquad d=-1$$

$$\pi: 2x+y-1=0$$

es $A=(1,2,1)$

$$s=\begin{cases} x-2y+z=1 \\ 2x+y-z=0 \end{cases}$$

$\pi \perp r$ passante per A

$\text{dist}(\pi, s)$

$$s=\begin{cases} x=t/3+1/3 \\ y=t \\ z=\frac{5t+2}{3} \end{cases} \qquad v_s=\begin{bmatrix} 1/3 \\ 1 \\ 5/3 \end{bmatrix}=\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

$$x=2t-z+1$$
$$2x+t-z=0$$

$$4t-2z+2+t-z=0$$

$$5t-3z+2=0 \qquad 3z=5t+2$$

$$v_r \cdot v_s=0$$

B sulla retta s

$$B=\left(\frac{t+1}{3};t;\frac{5t+2}{3}\right) \qquad \vec{AB}=\begin{bmatrix} \frac{t+1}{3}-1 \\ t-2 \\ \frac{5t+2}{3}-1 \end{bmatrix}=\begin{bmatrix} \frac{t-2}{3} \\ t-2 \\ \frac{5t-1}{3} \end{bmatrix}$$

$B = \left(\frac{\tau+1}{3} ; \tau ; \frac{5\tau+2}{3} \right) \qquad \vec{AB} = \begin{bmatrix} 3 \\ \tau - 2 \\ \frac{5\tau+2}{3} - 1 \end{bmatrix} = \begin{bmatrix} 3 \\ \tau - 2 \\ \frac{5\tau-1}{3} \end{bmatrix}$

$\vec{AB} \cdot \underline{v}_s = 0 \qquad \left(\frac{\tau-2}{3} \right) \cdot 1 + (\tau - 2) \cdot 3 + \left(\frac{5\tau-1}{3} \right) \cdot 5 = 0$

$\frac{\tau-2}{3} + 3\tau - 6 + \frac{25\tau - 5}{3} = 0$

$\tau - 2 + 9\tau - 18 + 25\tau - 5 = 0$

$35\tau = 25 \qquad \tau = \frac{5}{7}$

$B = \left(\left(\frac{5}{7} + 1 \right) \cdot \frac{1}{3} ; \frac{5}{7} ; \frac{25/7 + 2}{3} \right) = \left(\frac{4}{7} ; \frac{5}{7} ; \frac{13}{7} \right)$

la retta r passa per A e B

$\vec{AB} = \begin{bmatrix} \frac{\tau-2}{3} \\ \tau - 2 \\ \frac{5\tau-1}{3} \end{bmatrix} = \begin{bmatrix} -3/7 \\ -9/7 \\ 6/7 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$

$r: \begin{cases} x = 1 + \tau \\ y = 2 + 3\tau \\ z = 1 - 2\tau \end{cases}$

$\underline{v}_r = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \underline{u}$

$x + 3y - 2z + d = 0$

passa per A $\quad 1 + 6 - 2 + d = 0 \qquad d = -5$

$\pi : x + 3y - 2z - 5 = 0$

s e π sono incidenti?

$s \cap \pi = \{P\} \qquad d(s, \pi) = 0$

se s e π sono //?

$d(s, \pi) = d(Q, \pi)$

$s = \begin{cases} x - 2y + z = 1 \\ 2x + y - z = 0 \end{cases} \qquad \pi : x + 3y - 2z - 5 = 0$

metto a sistema

$\Big\{$

$x = 2y - z + 1$

$\begin{cases} 4y - 2z + 2 + y - z = 0 \\ 2y - z + 1 + 3y - 2z - 5 = 0 \end{cases}$

$$\begin{cases} 5y - 3z + 2 = 0 \\ 5y - 3z - 4 = 0 \end{cases} \quad \text{IMPOSSIBILE} \quad s \cap \pi = \emptyset$$

$$\underline{v}_s = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \qquad \underline{u} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \qquad \underline{v}_s \cdot \underline{u} = 0$$

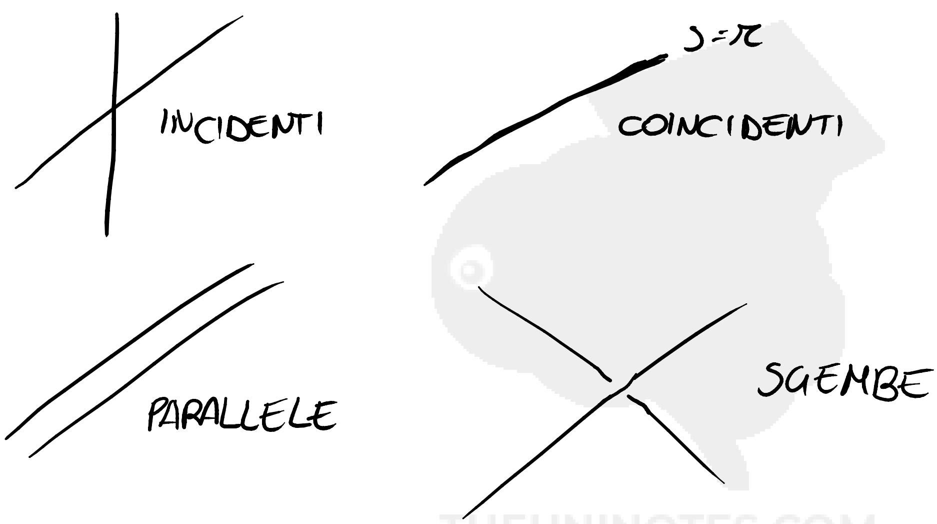
$$d(s,\pi) = d(\underset{\downarrow\ \text{su } \pi}{A}, \underset{\hookrightarrow\ \text{su } s}{B})$$

$$\|\overrightarrow{AB}\| = \sqrt{\frac{9}{49} + \frac{81}{49} + \frac{36}{49}} = \sqrt{\frac{126}{49}}$$

es

$$\pi = \begin{cases} x + z - y = 4 \\ 2x = 2 + y \end{cases} \qquad s = \begin{cases} z + y = 2 \\ x = 5 \end{cases}$$

posizione reciproca?



$$\pi: \begin{cases} x = t \\ y = 2t - 2 \\ z = 2t - 2 - t + 4 = t + 2 \end{cases} \qquad \underline{v}_\pi = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$s: \begin{cases} x = 5 \\ y = k \\ z = -k + 2 \end{cases} \qquad \underline{v}_s = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\underline{v}_\pi \not\parallel \underline{v}_s \qquad \underline{v}_\pi \neq \alpha\, \underline{v}_s$$

$\Rightarrow$ π e s non sono //

$\Rightarrow$ nemmeno coincidenti

$$\begin{cases} x = t = 5 \\ y = 2t - 2 = k \\ z = t + 2 = -k + 2 \end{cases} \qquad \begin{cases} t = 5 \\ 2t - 2 = k \\ t + \cancel{2} = -k + \cancel{2} \end{cases} \qquad \begin{matrix} t = 5 \\ k = 8 \\ 5 \neq -8 \end{matrix}$$

sistema impossibile

$\Rightarrow$ non sono incidenti

$\Rightarrow$ π e s sono SGHEMBE

$\underline{v_r} \cdot \underline{v_s} = 2-1 \neq 0$ NON $\perp$

$\text{dist}(s,r) = d(A,B)$

B generico $B=(t;2t-2;t+2)$

A generico $A=(5;k;-k+2)$

$$\vec{AB} = \begin{bmatrix} 5-t \\ k-2t+2 \\ -k+2-t-2 \end{bmatrix} = \begin{bmatrix} 5-t \\ k-2t+2 \\ -k-t \end{bmatrix}$$

$$\left.\begin{array}{l} \vec{AB}\cdot \underline{v_s} = 0 = 0(5-t)+1(k-2t+2)-1(-k-t) \\ \vec{AB}\cdot \underline{v_r} = 0 = 1(5-t)+2(k-2t+2)+1(-k-t) \end{array}\right\}$$

$$\begin{cases} k-2t+2+k+t=0 \\ 5-t+2k-4t+4-k-t=0 \end{cases} \qquad \begin{cases} 2k-t+2=0 \\ k-6t+9=0 \end{cases}$$

$$\begin{cases} t=2k+2 \\ k-12k-12+9=0 \end{cases} \qquad -11k=3 \qquad \begin{cases} t=16/11 \\ k=-\frac{3}{11} \end{cases}$$

$$\vec{AB} = \begin{bmatrix} 5-t \\ k-2t+2 \\ -k-t \end{bmatrix} = \begin{bmatrix} 5-16/11 \\ -\frac{3}{11}-\frac{32}{11}+2 \\ \frac{3}{11}-\frac{16}{11} \end{bmatrix} = \begin{bmatrix} \frac{39}{11} \\ -\frac{13}{11} \\ -\frac{13}{11} \end{bmatrix}$$

$$d(r,s) = \|\vec{AB}\| = \sqrt{\frac{39^2}{121}+\frac{13^2}{121}+\frac{13^2}{121}}$$

es5 $P=(3,4,1)$ $\pi: -x+3y+7z-2=0$

$$\underline{u} = \begin{bmatrix} -1 \\ 3 \\ 7 \end{bmatrix} = \underline{v_r}$$

$$r: \begin{cases} x = 3-t \\ y = 4+3t \\ z = 1+7t \end{cases}$$

cerco il punto A $\begin{cases} -x+3y+7z-2=0 \\ \begin{cases} x=3-t \\ y=4+3t \\ z=1+7t \end{cases} \end{cases}$

$$-(3-t)+3(4+3t)+7(1+7t)-2=0$$

$$-3+t+12+9t+7+49t-2=0$$

$$59t=-14 \qquad t=-\frac{14}{59}$$

$$A=\left(3+\frac{14}{59};\ 4-\frac{3\cdot 14}{59};\ 1-\frac{7\cdot 14}{59}\right) \qquad P=(3,4,1)$$

$$A = \left(3 + \frac{14}{59} ; 4 - \frac{3 \cdot 14}{59} ; 1 - \frac{7 \cdot 14}{59}\right) \qquad P = (3, 4, 1)$$

$$d(A,P) = |\overrightarrow{AP}| = \sqrt{\left(\frac{14}{59}\right)^2 + \left(\frac{3 \cdot 14}{59}\right)^2 + \left(\frac{7 \cdot 14}{59}\right)^2} =$$

$$= \frac{14}{59}\sqrt{1+9+49} = \frac{14}{59}\sqrt{59}$$