16/11/2020

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ESERCIZIO

$$f(x) = e^{2x^2} - \frac{1+\alpha x^2}{1+\beta x^2}$$

$$e^{2x^2} = 1 + 2x^2 + \frac{4x^4}{2} + \frac{(2x^2)^3}{3!} + o(x^6)$$

$$\frac{1+\alpha x^2}{1+\beta x^2} = (1+\alpha x^2)(1+\beta x^2)^{-1}$$

$$(1+t)^\alpha = 1 + \alpha t + \frac{\alpha(\alpha-1)}{2}t^2$$

$$(1+t)^{-1} = 1 - t + t^2 - t^3$$

$$(1+\alpha x^2)\left(1 - \beta x^2 + (\beta x^2)^2 - \beta(x^2)^3 + o(x^6)\right) =$$

$$= 1 + (\alpha-\beta)x^2 + (\beta^2 - \alpha\beta)x^4 + (\alpha\beta^2 - \beta^3)x^6 + o(x^6)$$

$$f(x) = \cancel{[2-(\alpha-\beta)]x^2} + [2 - \beta(\beta-\alpha)]x^4 + \left[\frac{4}{3} - \beta^2(\alpha-\beta)\right]x^6 + o(x^6)$$

visto che $x \to 0$ conta la potenza minore

$\alpha - \beta \neq 2 \Rightarrow f(x) \sim [2-(\alpha-\beta)]x^2$

infinitesimo del secondo ordine

$\alpha - \beta = 2, \beta \neq -1 \Rightarrow f(x) \sim (2+2\beta)x^4$

infinitesimo di ordine quarto

$\alpha - \beta = 2, \beta = 1 \Rightarrow f(x) \sim \left(\frac{4}{3} - 2\right)x^6 = -\frac{2}{3}x^6$

infinitesimo di ordine 6

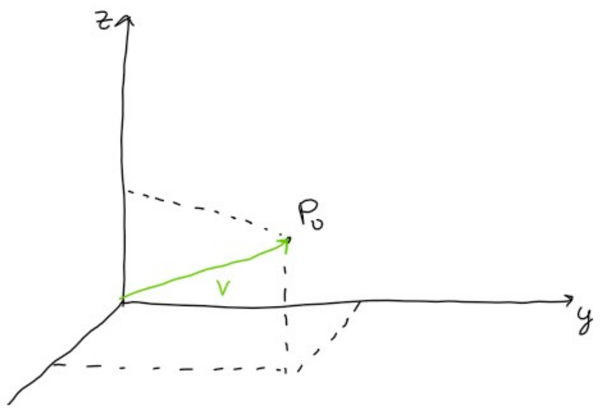
GEOMETRIA

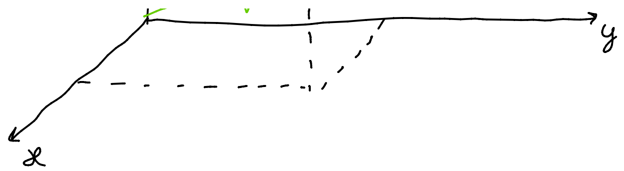
$\underline{v} \to$ segmento orientato con 3 caratteristiche:
- direzione
- verso
- lunghezza (modulo) $\|\underline{v}\|$ $\quad \|\underline{v}\| = 1 \to$ VERSORE

due vettori con stessa lunghezza, direzione e verso in due punti diversi dello spazio sono lo stesso vettore

$P_0 \equiv (x_0, y_0, z_0)$

$$\underline{v} = \overline{OP_0} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$$





$P_A \equiv (x_A, y_A, z_A)$
$P_B \equiv (x_B, y_B, z_B)$
$\underline{v} = \overline{P_A P_B} = \begin{bmatrix} x_B - x_A \\ y_B - y_A \\ z_B - z_A \end{bmatrix}$

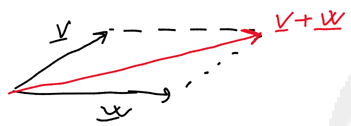
OPERAZIONI:

① PRODOTTO di un vettore per un numero reale λ

② SOMMA tra due vettori

① Dato $\underline{v}$ e il numero reale λ anche $\lambda \underline{v}$ è un vettore con:
- stessa direzione
- verso è lo stesso oppure opposto a seconda che λ sia positiva o negativa
- lunghezza $\|\lambda \underline{v}\| = |\lambda| \cdot \|\underline{v}\|$

② dati $\underline{v}$ e $\underline{w}$ $\quad \underline{v} + \underline{w}$ regola del parallelogramma



PROPRIETÀ SOMMA:
- commutativa $\underline{v} + \underline{w} = \underline{w} + \underline{v}$
- associativa $(\underline{u} + \underline{v}) + \underline{w} = \underline{u} + (\underline{v} + \underline{w})$
- esiste l'elemento neutro $\underline{z} : \underline{v} + \underline{z} = \underline{v} \quad \underline{0} \quad \underline{0} + \underline{v} = \underline{v}$
- esiste l'opposto $\forall \underline{v} \neq \underline{0} \ \exists \underline{o} : \underline{v} + \underline{o} = \underline{o} + \underline{v} = \underline{0}$
  $o = (-\underline{v})$

PRODOTTO:
- omog: $\lambda(\mu \underline{v}) = (\lambda \mu) \underline{v}$
- compatibilità con la norma $\|\lambda \underline{v}\| = |\lambda| \cdot \|\underline{v}\|$
- distributiva $\lambda(\underline{v} + \underline{w}) = \lambda \underline{v} + \lambda \underline{w}$
  $(\lambda + \mu)\underline{v} = \lambda \underline{v} + \mu \underline{v}$

$\Rightarrow (-\underline{v}) = (-1)\underline{v}$, $\lambda \underline{v} = \underline{0} \Rightarrow \lambda = 0$ opp $\underline{v} = \underline{0}$

Se verificano tutte le proprietà appartengono allo spazio vettoriale

$\underline{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \qquad \underline{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \qquad \lambda \in \mathbb{R}$

$\underline{v} + \underline{w} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \end{bmatrix}$ $\underline{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\lambda \underline{v} = \begin{bmatrix} \lambda v_1 \\ \lambda v_2 \\ \lambda v_3 \end{bmatrix}$ $-v = (-1)(\underline{v}) = \begin{bmatrix} -v_1 \\ -v_2 \\ -v_3 \end{bmatrix}$

$\|\underline{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$ $\frac{1}{\|\underline{v}\|}\underline{v}$ è un versore

VERSORI degli assi

$\underline{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \to$ asse x $\underline{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \to$ asse y $\underline{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \to$ asse z

$$\underline{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = a\underline{i} + b\underline{j} + c\underline{k}$$

- **PRODOTTO SCALARE**

prodotto tra due vettori che da come risultato uno scalare

$$\underline{v} \cdot \underline{w} = \|\underline{v}\| \cdot \|\underline{w}\| \cdot \cos\alpha$$

- commutativo $\underline{v} \cdot \underline{w} = \underline{w} \cdot \underline{v}$
- omog $(\lambda \underline{v}) \cdot \underline{w} = \underline{v} \cdot (\lambda \underline{w}) = \lambda(\underline{v} \cdot \underline{w})$
- distributiva $\underline{u} \cdot (\underline{v} + \underline{w}) = \underline{u} \cdot \underline{v} + \underline{u} \cdot \underline{w}$
- positività $\underline{u} \cdot \underline{u} \geq 0$ se $\underline{u} \cdot \underline{u} = 0 \iff \underline{u} = \underline{0}$
  $\to \|\underline{u}\| = \sqrt{\underline{u} \cdot \underline{u}}$

$\underline{v} \cdot \underline{w} = 0 \iff \underline{v} \perp \underline{w}$ $\underline{v}$ e $\underline{w} \neq \underline{0}$

PROPRIETÀ

se $v = v_1\underline{i} + v_2\underline{j} + v_3\underline{k}$ e $w = w_1\underline{i} + w_2\underline{j} + w_3\underline{k}$

$\Rightarrow \underline{v} \cdot \underline{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$

DIM

$\underline{v} \cdot \underline{w} = (v_1\underline{i} + v_2\underline{j} + v_3\underline{k}) \cdot (w_1\underline{i} + w_2\underline{j} + w_3\underline{k}) =$

$= v_1 w_1(\underline{i} \cdot \underline{i}) + \cancel{v_1 w_2(\underline{i} \cdot \underline{j})} + \cancel{v_1 w_3(\underline{i} \cdot \underline{k})} + \cancel{(v_2 w_1)(\underline{j} \cdot \underline{i})} + v_2 w_2(\underline{j} \cdot \underline{j})$
$+ \cancel{v_2 w_3(\underline{j} \cdot \underline{k})} + \cancel{v_3 w_1(\underline{k} \cdot \underline{i})} + \cancel{v_3 w_2(\underline{k} \cdot \underline{j})} + v_3 w_3(\underline{k} \cdot \underline{k})$

versori sono perpendicolari quindi il prodotto scalare è $= 0$

$= v_1 w_1 + v_2 w_2 + v_3 w_3$

PRODOTTO VETTORIALE

$\underline{v}$ e $\underline{w}$ $\underline{v} \times \underline{w}$ è un vettore

- direzione perpendicolare a $\underline{v}$ e $\underline{w}$
- verso t.c. $v, w, v \times w$ formano terna destr.

- direzione perpendicolare a $\underline{v}$ e $\underline{w}$
- verso t.c. $\underline{v}, \underline{w}, \underline{v} \times \underline{w}$ formano terna destr.
- lunghezza $\|\underline{v} \times \underline{w}\| = \|\underline{v}\| \cdot \|\underline{w}\| \cdot \sin\alpha$

PROPRIETA'

- anticommutativa $\underline{v} \times \underline{w} = -\underline{w} \times \underline{v}$
- omog $(\lambda \underline{v}) \times \underline{w} = \underline{v} \times (\lambda \underline{w}) = \lambda (\underline{v} \times \underline{w})$
- distributiva $\underline{u} \times (\underline{v} + \underline{w}) = \underline{u} \times \underline{v} + \underline{u} \times \underline{w}$

$\underline{v} \times \underline{w} = \underline{0}$ e $\underline{v}, \underline{w} \neq \underline{0} \implies \underline{v} \parallel \underline{w}$ $(\underline{v} = \lambda \underline{w})$

$\underline{v} = v_1 \underline{i} + v_2 \underline{j} + v_3 \underline{k}$ $\quad \underline{w} = w_1 \underline{i} + w_2 \underline{j} + w_3 \underline{k}$

$v \times w = (v_2 w_3 - v_3 w_2)\underline{i} + (v_3 w_1 - v_1 w_3)\underline{j} + (v_1 w_2 - v_2 w_1)\underline{k}$

DIM

$\underline{v} \times \underline{w} = (v_1 \underline{i} + v_2 \underline{j} + v_3 \underline{k}) \times (w_1 \underline{i} + w_2 \underline{j} + w_3 \underline{k}) =$

$= \cancel{v_1 w_1 (\underline{i} \times \underline{i})} + v_1 w_2 (\underbrace{\underline{i} \times \underline{j}}_{k}) + v_1 w_3 (\underbrace{\underline{i} \times \underline{k}}_{-j})$

$+ v_2 w_1 (\underbrace{\underline{j} \times \underline{i}}_{-k}) + \cancel{v_2 w_2 (\underline{j} \times \underline{j})} + v_2 w_3 (\underbrace{\underline{j} \times \underline{k}}_{i})$

$+ v_3 w_1 (\underbrace{\underline{k} \times \underline{i}}_{j}) + v_3 w_2 (\underbrace{\underline{k} \times \underline{j}}_{-i}) + \cancel{v_3 w_3 (\underline{k} \times \underline{k})}$

prodotto vettoriale di vettori // fa zero

per regola della mano destra

$= (v_2 w_3 - v_3 w_3)\underline{i} + (v_3 w_1 - v_1 w_3)\underline{j} + (v_1 w_2 + v_2 w_1)\underline{k}$

$$\underline{v} \times \underline{w} = \det \begin{bmatrix} i & j & k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}$$

es

$P_1 = (1, 2, -1)$ $\quad P_2 \equiv (3, 4, 4)$ $\quad P_3 \equiv (4, 2, 0)$

$A_{TR} = ?$

$\|\bar{v} \times \bar{w}\| = \|\bar{v}\| \cdot \|\bar{w}\| \sin\alpha \rightarrow$ area del parallelogramma

$A_{TR} = \frac{1}{2} \|\overrightarrow{P_1 P_2} \times \overrightarrow{P_1 P_3}\| = \frac{1}{2}$

$P_1 P_2 = 2\bar{i} + 2j + 5\bar{k}$
$P_1 P_3 = 3\bar{i} + \bar{k}$

$P_1 P_2 \times P_1 P_3 = -5\bar{i} + 13j - 6k$

$A = \frac{1}{2} \sqrt{4 + 169 + 36} = \frac{1}{2} \sqrt{209}$

PRODOTTO MISTO

$\bar{u}, \bar{v}, \bar{w}$ è $\bar{u} \cdot (\bar{v} \times \bar{w})$ [si fa prima il prodotto vett.]

PROPR.

- ciclico: $\bar{u} \cdot \bar{v} \times \bar{w} = \bar{v} \cdot \bar{u} \times \bar{w} = \bar{w} \cdot \bar{u} \times \bar{v}$

vett.]

PROPR.

- ciclico: $\bar{u} \cdot \bar{v} \times \bar{w} = \bar{v} \cdot \bar{u} \times \bar{w} = \bar{w} \cdot \bar{u} \times \bar{v}$
- $u \cdot v \times w = 0 \iff u, v$ e w sono complanari

$|u \cdot v \times w|$ = volume del parallelepipedo

$P_1 \equiv (1,1,0)$ $P_2 \equiv (2,3,4)$ $P_3 \equiv (2,2,4)$ $P_4 \equiv (3,1,-1)$

V della piramide

$P_1 \to$ campo base

$$V_P = \frac{1}{6} |\vec{P_1P_2} \cdot \vec{P_1P_3} \times \vec{P_1P_4}|$$

$P_1P_3 = 1\bar{i} + 1\bar{j} + 3\bar{k}$

$P_1P_4 = 2\bar{i} - \bar{k}$

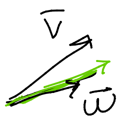
$\vec{P_1P_3} \times \vec{P_1P_4} = -\bar{i} + 7\bar{j} - 2\bar{k}$

$\vec{P_1P_2} = \bar{i} + 2\bar{j} + 4\bar{k}$

$P_1P_2 \cdot P_1P_3 \times P_1P_4 = -1 + 14 - 8 = 5$

$$V = \frac{5}{6}$$

PRODOTTO SCALARE legato alla proiezione ortogonale



$\bar{u}$ = proiezione di v nella direzione di w

$\bar{u} = \bar{w}\lambda$ (α acuto, $\lambda > 0$ se α ottuso $\lambda < 0$)

se α acuto $\|\bar{u}\| = \|\bar{v}\| \cos\alpha = \dfrac{\bar{v} \cdot \bar{w}}{\|\bar{w}\|}$ $\quad u = \dfrac{\bar{v} \cdot \bar{w}}{\|\bar{w}\|} \cdot \dfrac{\bar{w}}{\|\bar{w}\|}$

α ottuso $\|\bar{u}\| = \|\bar{v}\|(-\cos\alpha)$

$$\bar{u} = \frac{\bar{v} \cdot \bar{w}}{\|\bar{w}\|} \cdot \frac{\bar{w}}{\|\bar{w}\|} = \underbrace{\left(\frac{\bar{v} \cdot \bar{w}}{\|\bar{w}\|^2}\right)}_{\text{scalare}} \bar{w} = \underbrace{\left(\frac{\bar{v} \cdot \bar{w}}{\bar{w} \cdot \bar{w}}\right)}_{\lambda} \cdot \bar{w}$$

INDIVIDUARE UN PIANO NELLO SPAZIO

- 3 punti non allineati
- 1 punto e 2 vettori
- 1 punto e 1 vettore perpendicolare

$P_0 \equiv (x_0, y_0, z_0)$ $\quad \bar{v} = a\bar{i} + b\bar{j} + c\bar{k}$

prendo un punto $P \in$ al piano $\iff \vec{P_0P} \perp \bar{v}$

$\bar{v} \cdot \vec{P_0P} = 0$

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

$ax + by + cz = d$

↳ $ax_0 + by_0 + cz_0$

$$ax + by + cz = d$$

$$\hookrightarrow ax_0 + by_0 + cz_0$$

$P_0 \equiv (2, 3, -1) \perp \bar{v} = 3\bar{i} + \bar{j} - \bar{k}$

$$\begin{array}{ccccccc} 3x & + & y & - & z & = & 10 \\ | & & | & & | & & \\ 2 & & 3 & & -1 & & \end{array}$$

$P_1 \equiv (2, 1, 3) \quad P_2 \equiv (3, 3, 0) \quad P_3 \equiv (4, 2, 5)$

$\overrightarrow{P_1P_2} = \bar{i} + 2\bar{j} - 3\bar{k}$

$\overrightarrow{P_1P_3} = 2\bar{i} + \bar{j} + 2\bar{k}$

$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = 7\bar{i} - 8\bar{j} - 3\bar{k}$

$$\begin{array}{ccccccc} 7x & - & 8y & - & 3z & = & -3 \\ | & & | & & | & & \\ 2 & & 1 & & 3 & & \end{array}$$